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LOCAL SOLVABILITY OF ONE-DIMENSIONAL PROBLEM OF THERMOVISCOELASTICITY

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ABSTRACT

The local solvability of the initial-boundary value problem in spaces of summable functions for some one-dimensional system of equations of thermoviscoelasticity is established. The nonlinearities in equations are determined by the difference between Lagrangian and Eulerian coordinates. The coercive approach implies a necessary condition on the initial data.

1.

The purpose of this paper is to establish a local solvability of the initial-boundary value problem associated with the equations of one-dimensional physical linear thermoviscoelasticity

(1)
$$u''_{tt} - k_1 u''_{xx} - k_2[(1+u'_x)^{-1}u''_{tx}]'_x - k_3\theta'_x = f(t,x)$$
$$(t,x) \in Q = [0,t_0] \times [0,1], \quad k_i > 0;$$

(2)
$$\theta'_t - k_4[(1+u'_x)^{-1}\theta'_x]'_x - k_3\theta u''_{tx} + k_2(u''_{tx})^2 = \varphi(t,x), \quad (t,x) \in Q, \quad k_4 > 0;$$

(3)
$$u(0,x) = u_0(x), \quad u'_t(0,x) = u_1(x) \quad (0 \le x \le 1),$$
$$u(t,0) = u(t,1) = 0 \quad (0 \le t \le t_0);$$

(4)
$$\theta(0,x) = \theta_0(x) \quad (0 \le x \le 1), \quad \theta(t,0) = \theta(t,1) = 0 \quad (0 \le t \le t_0).$$

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Here $u(t,x)$ and $\theta(t,x)$ denote displacement and temperature of a medium. The equations (1) and (2) are the conservation laws of impulse and energy in Lagrangian coordinates, respectively. The model (1)–(4) is constructed under the following conditions: 1) the stress tensor (one-dimensional) is a linear combination of Almansi strain tensor and the tensor of velocity of deformations; 2) the internal energy is a linear combination of the temperature and a square of the strain; 3) the Fourier law for a heat conduction is valid.

The cases of thermoelasticity ($k_2 = 0$) and viscoelasticity ($\theta = 0$) were considered in [3] and [4], [5]. Here we establish similar results in the general case.

2.

A solution of problem (1)–(4) is defined to be a pair of functions $u(t,x)$, $\theta(t,x)$ having all (generalized) derivatives contained in the equations (1)–(2) in $L_p = L_p(Q)$, $1 < p < +\infty$ and satisfying equations (1)–(2) and conditions (3)–(4). It is supposed in addition that

$$1 + u_x'(t,x) > 0, \quad (t,x) \in Q. \tag{5}$$

Let $W_p^{k,m}$ (see [1]) be a Banach space of the functions on Q having in $L_p(Q)$ all derivatives up to order k with respect to t and up to order m with respect to x. Let W_p^k be a Banach space of the functions $\varphi(x)$, $0 \le x \le 1$, having all derivatives up to order k (not necessary integer) in $L_p[0,1]$. We denote by $\|\cdot\|_{k,m}$, $|\cdot|_k$, $\|\cdot\|_0$ and $|\cdot|_0$ the norms in $W_p^{k,m}, W_p^k, L_p(Q)$ and $L_p[0,1]$, respectively. And finally, we use the notation

$$\overset{0}{W}{}_p^k = \{\varphi(x) : \varphi(x) \in W_p^k, \varphi(0) = \varphi(1) = 0\}.$$

THEOREM 1: *Let $f, \varphi \in L_p$ for some $p \in (3,\infty)$. Let $u_0 \in \overset{0}{W}{}_p^2$, $u_1 \in \overset{0}{W}{}_p^{2-2/p}$, $\theta_0 \in \overset{0}{W}{}_p^{2-2/p}$. Let*

$$1 + u_0'(x) > 0, \quad 0 \le x \le 1. \tag{6}$$

Then for sufficiently small t_0 there exists a unique solution of the problem (1)–(4).

Note that the necessity of our conditions on initial data follows from the properties of solutions (see below) of problem (1)–(4) according to results on linear parabolic initial-boundary value problems from [2].

3.

Now we shall give a sketch of the proof of Theorem 1. Consider a certain set S_R in $W_p^{1,2}$. Take an arbitrary $\theta \in S_R$ and find a solution u of problem (1), (3). Then find a solution $\tilde{\theta}$ of problem (2), (4) for this u. Denote by $\mathfrak{N}$ the operator assigning $\tilde{\theta}$ to θ. We shall show that the operator $\mathfrak{N}$ has a unique fixed point θ in S_R for sufficiently small t_0 and, hence, the pair θ, u (u being solution of problem (1), (3) for this θ) is a unique solution of problem (1)–(4). The auxiliary results on solvability and estimates for solutions of a certain linear problems are established in §4, solvability and estimates of problems (1), (3) and (2), (4) are obtained in §5 and §6. The direct proof of Theorem 1 is in §7.

4.

Let us consider at first the linear problem:

$$v'_t - k_2[(1+u'_0(x))^{-1}v'_x]'_x = z(t,x), \quad (t,x) \in Q; \tag{7}$$

$$v(0,x) = u_1(x), \quad 0 \le x \le 1; \quad v(t,0) = v(t,1) = 0, \quad 0 \le t \le t_0. \tag{8}$$

From [2] and [6] it follows that the problem (7)–(8) has a unique solution for any $z \in L_p$ and $u_1 \in \overset{0}{W}{}_p^{2-2/p}$ ($u_0(x)$ satisfies the conditions of Theorem 1) and the estimate

$$\|v\|_{1,2} + \max_{0\le t\le t_0} |v(t,x)|_{2-2/p} \le M_1(\|z\|_0 + |u_1|_{2-2/p}) \tag{9}$$

holds. Here M_1 depends on $u_0(x)$.

The continuous embedding (see [1])

$$W_p^{2-2/p} \subset C^1[0,1] \quad (3 < p < \infty) \tag{10}$$

and (9) imply the inequality

$$\max_{(t,x)\in Q} |v'_x(t,x)| \le M_2(\|z\|_0 + |u_1|_{2-2/p}). \tag{11}$$

Choosing $p_1 \in (3,p)$ and using inequalities (9) and (10) for the chosen p_1 it is easy to show that inequality

$$\max_{(t,x)\in Q} |\tilde{v}'_x(t,x) - \tilde{v}'_x(0,x)| \le M_3 t_0^{\gamma_0}(\|\tilde{v}\|_{1,2} + |\tilde{v}(0,x)|_{2-2/p}), \tag{12}$$

$$0 < \gamma_0 < \frac{p-3}{3},$$

is valid for arbitrary $\tilde{v}(t,x) \in W_p^{1,2}$.

This and (9) yield

$$\max_{(t,x)\in Q} |v_x'(t,x) - u_1'(x)| \le M_4 t_0^{\gamma_0}(\|z\|_0 + |u_1|_{2-2/p}). \tag{13}$$

Now consider the problem

$$u_{tt}'' - k_1 u_{xx}'' - k_2[(1+u_0'(x))^{-1}u_{tx}'']_x' = w(t,x), \quad (t,x) \in Q; \tag{14}$$

$$\begin{gathered} u(0,x) = \tilde{u}_0(x), \quad u_t'(0,x) = u_1(x), \quad 0 \le x \le 1; \\ u(t,0) = u(t,1) = 0, \quad 0 \le t \le t_0. \end{gathered} \tag{15}$$

Denoting $u_t' = v$, representing equation (14) in the form (7) with

$$z = w + k_1 \int_0^t v_{xx}''(s,x)ds + u_0'' \tag{16}$$

and inversing the operator generated by problem (7)–(8) in L_p, we obtain the following result.

LEMMA 1: *For any $w \in L_p$, $\tilde{u}_0 \in \overset{\circ}{W}{}_p^{2-2/p}$ and u_0, u_1 satisfying the conditions of Theorem 1 the problem (14)–(15) has a unique solution and the estimate*

$$|||u||| \le M_5 \mathfrak{M}(\tilde{u}_0, u_1, w) \tag{17}$$

holds. Here

$$|||u||| = \|u_{tt}''\|_0 + \|u_{txx}'''\|_0 + \|u_{xx}''\|_0 + \|u\|_0,$$

$$\mathfrak{M}(\tilde{u}_0, u_1, w) = |\tilde{u}_0|_2 + |u_1|_{2-2/p} + \|w\|_0.$$

In addition to Lemma 1 we have

LEMMA 2: *Under the conditions of Lemma 1 for a solution of problem (14)–(15) the inclusions*

$$u, u_x', u_{tx}'' \in C[Q] \tag{18}$$

and estimates

$$\max_{(t,x)\in Q} |u_x'(t,x) - u_0'(x)| \le M_6 t_0^{\gamma_1} \mathfrak{M}(\tilde{u}_0, u_1, w), \quad \gamma_1 = \gamma_0 + 1; \tag{19}$$

$$\max_{(t,x)\in Q} |u''_{xt}(t,x) - u'_1(x)| \le M_7 t_0^{\gamma_2} \mathfrak{M}(\tilde{u}_0, u_1, w), \quad \gamma_2 = \gamma_0; \tag{20}$$

$$\max_{0\le t\le t_0} |u''_{xx}(t,x) - u''_0(x)|_0 \le M_8 t_0^{\gamma_3} \mathfrak{M}(\tilde{u}_0, u_1, w), \quad \gamma_3 = 1 - 1/p \tag{21}$$

are valid.

Proof of Lemma 2: From the definition of a solution of problem (14)–(15) it follows that z defined by (16) belongs to L_p. Then $v = u'_t$ is a solution of problem (7)–(8) for this z. Therefore, $u''_{tx} = v'_x$ is continuous and the estimate (20) follows from (13).

Since

$$u'_x(t,x) - u'_x(0,x) = \int_0^t u''_{tx}(s,x)ds, \tag{22}$$

the estimate (19) follows from (20). In order to obtain inequality (21) we have to differentiate (22) by x and apply Hölder's inequality and estimate (17).

Lemma 2 is proved. ∎

Let u^1 and u^2 be solutions of problem (14)–(15) for $w = w^1$ and $w = w^2$ respectively.

LEMMA 3: *Let $w^i \in L_p (i = 1,2)$ and $u_0, \tilde{u}_0, u_1$ satisfy the conditions of Lemma 2. Then there are inequalities*

$$\max_{(t,x)\in Q} \left|\frac{\partial}{\partial x}u^1(t,x) - \frac{\partial}{\partial x}u^2(t,x)\right| \le M_9 t_0^{\gamma_4} \|w^1 - w^2\|_0, \quad \gamma_4 = \gamma_0 + 1; \tag{23}$$

$$\max_{(t,x)\in Q} \left|\frac{\partial^2}{\partial t \partial x}u^1(t,x) - \frac{\partial^2}{\partial t \partial x}u^2(t,x)\right| \le M_{10} t^{\gamma_5} \|w^1 - w^2\|_0, \quad \gamma_5 = \gamma_0; \tag{24}$$

$$\max_{0\le t\le t_0} \left|\frac{\partial^2}{\partial x^2}u^1(t,x) - \frac{\partial^2}{\partial x^2}u^2(t,x)\right|_0 \le M_{11} t_0^{\gamma_6} \|w^1 - w^2\|_0, \quad \gamma_6 = 1 - 1/p. \tag{25}$$

The proof of the lemma easily follows from the linearity of problem (14)-(15) by means of inequalities (19)–(21).

5.

Let us assume that θ is known and let us study solvability of problem (1),(3).

THEOREM 2: *Let f, u_0, u_1 satisfy the conditions of Theorem 1 and $\theta \in W_p^{0,1}$. Let*

$$\mathfrak{M}(u_0, u_1, f) \le R_1, \quad \|\theta'_x\| \le R_0. \tag{26}$$

Then there exists sufficiently small t_0 such that the problem (1), (3) has a unique solution and the estimates

$$|||u||| \le M_{12}; \tag{27}$$

$$\max_{(t,x)\in Q} |u'_x(t,x) - u'_0(x)| \le M_{13} t_0^{\gamma_7}, \quad \gamma_7 = \gamma_1; \tag{28}$$

$$\max_{(t,x)\in Q} |u''_{tx}(t,x) - u'_1(x)| \le M_{14} t_0^{\gamma_8}, \quad \gamma_8 = \gamma_2 \tag{29}$$

hold. Here $M_i = M_i(R_1, R_0)$, $t_0 \ne t_0(R_0)$.

Proof of Theorem 2: Let us represent problem (1), (3) in the form

$$\begin{aligned} u''_{tt} - k_2[(1+u'_0(x))^{-1}u''_{tx}]'_x - k_1 u''_{xx} = f + k_3\theta'_x \\ +k_2[u''_0 - u''_{xx}](1+u'_0)^{-1}(1+u'_x)^{-1}u''_{tx} - k_2(u'_0 - u'_x)(1+u'_x)^{-2}(1+u'_0)^{-1}u''_{xx}u''_{tx} \\ -k_2(u'_0 - u'_x)(1+u'_x)^{-1}(1+u'_0)^{-1}u''_0 u''_{tt} + k_2(u'_0 - u'_x)(1+u'_x)^{-1}(1+u'_0)^{-1}u'''_{txx}; \end{aligned} \tag{30}$$

$$u(0,x) = u_0(x), \quad u'_t(0,x) = u_1(x), \quad u(t,0) = u(t,1) = 0. \tag{31}$$

Denote the right-hand side of (30) by w. Let

$$u = L^{-1}(u_0, u_1, w) \tag{32}$$

where L^{-1} is the operator defined by the solution of (14)–(15).

Using relations (30)–(31) and L^{-1} we reduce problem (1), (3) to the equivalent integral equation

$$w = K(w) \tag{33}$$

where

$$K(w) = f + k_3\theta + k_2[K_1(w) - K_2(w) - K_3(w) + K_4(w)]; \tag{34}$$

$$K_1(w) = (u_0'' - u_{xx}'')(1+u_x')^{-2}(1+u_0')^{-1}u_{tx}'' \quad (\equiv F_1(u)); \tag{35}$$

$$K_2(w) = (u_0' - u_x')(1+u_x')^{-2}(1+u_0')^{-1}u_{xx}''u_{tx}'' \quad (\equiv F_2(u)); \tag{36}$$

$$K_3(w) = (u_0' - u_x')(1+u_x')^{-1}(1+u_0')^{-2}u_{tx}''u_0'' \quad (\equiv F_3(u)); \tag{37}$$

$$K_4(w) = (u_0' - u_x')(1+u_x')^{-1}(1+u_0')^{-1}u_{txx}''' \quad (\equiv F_4(u)). \tag{38}$$

We have in mind

$$u = L^{-1}(u_0, u_1, w). \tag{39}$$

We establish unique solvability of (33) by means of the fixed-point theorem for contractions. Let

$$S_{R_2} = \{w : w \in L_p, \|w\|_0 \le R_2\}.$$

LEMMA 5: *Let R_2 be sufficiently large and t_0 be sufficiently small. Then we have the inclusion*

$$KS_{R_2} \subset S_{R_2}. \tag{40}$$

LEMMA 6: *Under conditions of Lemma 5 for any w_1, $w_2 \in S_{R_2}$ we have the inequality*

$$\|K(w_1) - K(w_2)\|_0 \le M_{15}t_0^{\gamma_9}\|w_1 - w_2\|_0, \quad \gamma_9 > 0. \tag{41}$$

Proof of Lemma 5: At first let us prove that for any

$$t_0 \le [\frac{m_0}{2}(R_2 + 2R_1)]^{-\gamma_1} \quad (m_0 = \min_{0\le x\le 1}|1 + u_0'(x)|), \tag{42}$$

there is the inequality

$$|1 + u_x'(t,x)|^{-1} \le 2m_0^{-1}. \tag{43}$$

Really, from (39) and (20), (26) for $f = w$ we get

$$|1 + u'_x(t,x)| \geq |1 + u'_0(x)| - \max_{t,x} |u'_x(t,x) - u'_0(x)|$$
$$\geq m_0 - M_6 t_0^{\gamma_1} \mathfrak{M}(u_0, u_1, w) \geq m_0 - M_6 t_0^{\gamma_1}(R_2 + 2R_1).$$

This implies (43) in view of (42). Next, estimating $K(w)$ and using (34) and (26), we have

$$\|K(w)\|_0 \leq \|f\|_0 + k_3\|\theta'_x\|_0 + k_3 \sum_{i=1}^{4} \|K_i(w)\|_0 \leq R_1 + k_3 R_0 + \sum_{i=1}^{4} \|K_i(w)\|_0.$$

Estimating the $\|\cdot\|_0$ -norm of the term containing the highest derivative and the uniform norm of the other terms and using inequalities (17), (20), (26) and (43), we obtain

$$\|K_4(w)\|_0 \leq M_{16} t_0^{\gamma_1} \mathfrak{M}^2(u_0, u_1, w) \leq M_{17} t_0^{\delta_1}(R_2 + R_1)^2.$$

In a similar way we get

$$\|K_i(w)\|_0 \leq M_{18} t_0^{\gamma_1}(R_2 + R_1), \quad i \neq 4.$$

From the latter inequalities it follows that

$$\|K(w)\|_0 \leq R_1 + k_3 R_0 + M_{19} t_0^{\gamma_1}(\|w\|_0^2 + R_1^2). \tag{44}$$

Choosing $R_2 > R_1 + k_3 R_0$ and t_0 sufficiently small, we get (40). Lemma 4 is proved. ∎

Proof of Lemma 5: It is sufficient to prove the inequalities

$$\|K_i(w_1) - K_i(w_2)\|_0 \leq M_{20} t_0^{\gamma_9} \|w_1 - w_2\|_0, \quad 1 \leq i \leq 4. \tag{45}$$

We consider only the case $i = 4$. It is obvious that

$$K_4(w_1) - K_4(w_2) = (u'_0 - u'_x)(1 + u'_0)^{-1}(1 + u'_x)^{-1}[u'''_{txx} - v'''_{txx}]$$
$$+(u'_0 - u'_x)(1 + u'_0)^{-1}(1 + u'_x)^{-1}(1 + v'_x)^{-2}(v'_x - u'_x)v'''_{txx}$$
$$+(u'_x - v'_x)(1 + u'_0)^{-1}(1 + v'_x)^{-1} v'''_{txx}.$$

Here $u = L^{-1}(u_0, u_1, w_1)$, $v = L^{-1}(u_0, u_1, w_2)$. Using estimates (17), (14), (23), (43) and estimating the $\|\cdot\|_0$ norm of the highest derivatives and the uniform norm of the lower derivatives, we obtain (45) for $i = 4$. Lemma 5 is proved. ∎

From Lemmas 4 and 5 it follows that the conditions of the fixed-point theorem for contractions with respect to (33) are satisfied for sufficiently large R_2 and sufficiently small t_0. Therefore, the equation (33) and consequently the problem (1), (3) has a unique solution. Their solutions are connected by means of formula (39).

Let us establish the estimates (27)–(29). From the estimate (44) it follows that for the solution w of (33) there is the inequality

$$\|w\|_0 \leq R_1 + k_3 R_0 + M_{21} t_0^{\gamma_1} (\|w\|_0^2 + R_1^2). \tag{46}$$

From this for $0 < t_0 < (2M_{21}R_2)^{-\gamma_1}$ we obtain the result.

Taking into account formula (39) and Lemma 2 we get (27)–(29). Theorem 2 is proved. ∎

Let $u(t,x)$ be a solution of problem (1), (3) and $\tilde{u}(t,x)$ be a solution of problem (1), (3) for $\theta = \tilde{\theta}$.

THEOREM 3: *Let $u_0, u_1, f, \theta, \tilde{\theta}$ satisfy the conditions of Theorem 2. Then we have inequalities*

$$|||u - \tilde{u}||| \leq M_{23} \|\theta'_x - \tilde{\theta}'_x\|_0; \tag{47}$$

$$\max_{(t,x)\in Q} |u'_x(t,x) - \tilde{u}'_x(t,x)| \leq M_{24} \|\theta'_x - \tilde{\theta}'_x\|_0; \tag{48}$$

$$\max_{(t,x)\in Q} |u''_{tx}(t,x) - \tilde{u}''_{tx}(t,x)| \leq M_{25} \|\theta'_x - \tilde{\theta}'_x\|_0. \tag{49}$$

Here $m_i = M_i(R_1, R_0)$.

Proof of Theorem 3: Let $z = u - \tilde{u}$. Then

$$z''_{tt} - k_2[(1+u'_0)^{-1} z''_{tx}]'_x - k_1 z''_{xx} = k_3[\theta'_x - \tilde{\theta}'_x] + k_2 \sum_{i=1}^{4} P_i(u,\tilde{u}) \tag{50}$$

where $P_i(u,\tilde{u}) = F_i(u) - F_i(\tilde{u})$ and F_i are defined by formulas (35)–(39).

At first we show that

$$\|P_i(u,\tilde{u})\|_0 \le M_{26} t_0^{\gamma_{10}} |||u-\tilde{u}|||, \quad i=1,2,3,4, \quad \gamma_{10}>0. \tag{51}$$

It is easy to see that

$$\begin{aligned} P_1(u,\tilde{u}) &= (\tilde{u}''_{xx}-u''_{xx})(1+u'_x)^{-1}(1+u'_0)^{-1}u''_{tx} \\ &+(u''_0-u''_{xx})(\tilde{u}'_x-u'_x)(1+u'_x)^{-1}(1+\tilde{u}'_x)^{-1}(1+u'_0)^{-1}u''_{tx} \\ &+(u''_0-\tilde{u}''_{xx})(1+\tilde{u}'_x)^{-1}(1+u'_0)^{-1}(u''_{tx}-\tilde{u}''_{tx}) = \sum_{i=1}^{3} Q_i. \end{aligned}$$

From the Newton–Leibniz formula and Hölder's inequality it follows that

$$|u''_{xx}(t,x)-\tilde{u}''_{xx}(t,x)| \le \int_0^t |u'''_{txx}(s,x)-\tilde{u}'''_{txx}(s,x)|ds \le t_0^{1-1/p}\|u'''_{txx}-\tilde{u}'''_{txx}\|_0.$$

From this and from (27), (43) we obtain

$$\|Q_1\|_0 \le M_{27} t_0 |||u-\tilde{u}|||. \tag{52}$$

From inequalities (19), (21), (27) and (43) by means of Sobolov's embedding theorems we get

$$\|Q_2+Q_3\|_0 \le M_{28} t_0^{\gamma_{11}}\big(\max_{(t,x)\in Q}|u'_x-\tilde{u}'_x| + \max_{(t,x)\in Q}|u''_{tx}-\tilde{u}''_{tx}|\big) \le M_{29} t_0^{\gamma_{11}} |||u-\tilde{u}|||. \tag{53}$$

Using (52) and (53), we get (51) for $i=1$.

The other inequalities (51) are proved in a similar way.

Applying estimates (17) to the linear equation (50) and taking into account (51) we obtain

$$|||u-\tilde{u}||| \le M_{30}(\|\theta'_x-\tilde{\theta}'_x\|_0 + t_0^{\gamma_{10}}|||u-\tilde{u}|||).$$

From this for small t_0 we obtain the inequality (47). Inequalities (48)–(49) easily follow from (47). Theorem 3 is proved. ∎

6.

Let us suppose that $u(t,x)$ is known and consider the linear problem (2), (4).

THEOREM 4: *Let $u_0(x)$ satisfy the condition of Theorem 1. Let $u(t,x)$ satisfy the conditions (27)–(29) for certain M_i and γ_i. Let $\varphi \in L_p, \theta_0 \in W_p^{2-2/p}$ and*

$$\max(\|\varphi\|_0, |\theta_0|_{2-2/p}) \le R_3 \tag{54}$$

Then for sufficiently small t_0 the problem (2), (4) has a unique solution and the estimate

$$\|\theta\|_{1,2} \le M_{31}(R_3+1) \tag{55}$$

holds. Here M_{31} depends on M_i $(i = 12, 13, 14)$.

Proof of Theorem 4: From the estimates (27)–(29) and (6) it follows that for sufficiently small t_0 the functions u'_x and u''_{tx} are continuous, uniformly bounded on Q (with respect to the functions u satisfying (27)–(29)) and

$$0 < m_1 \le |1 + u'_x(t,x)|^{-1} \le 2m_0^{-1} \tag{56}$$

Taking into account (55) and using theorem 9.1 from [2] we get unique solvability of the problem (2), (4) and the estimate

$$\|\theta\|_{1,2} \le M_{32}(\|\varphi + k_3(u''_{tx})^2\|_0 + |\theta_0|_{2-2/p}) \le M_{33}R_3 + M_{34} \tag{57}$$

where M_i $(i = 32, 33, 34)$ depend on M_i $(i = 12, 13, 14)$. Theorem 4 is proved. ∎

The estimate (55) by means of (9) and (10) implies the continuity of the functions θ and θ'_x on Q and the inequality

$$\max_{(t,x)\in Q} |\theta(t,x)| + \max_{(t,x)\in Q} |\theta'_x| \le M_{35}(R_3+1) + |\theta_0|_{2-2/p} \tag{58}$$

for $3 < p < +\infty$. Here M_{35} is similar to M_{34}.

Let now θ be a solution of problem (2), (4) and $\tilde{\theta}$ a solution of problem (2), (4) for $u = \tilde{u}$.

THEOREM 5: *Let $u_0, u, \tilde{u}, \theta, \theta_0$ satisfy the conditions of the Theorem 4. Then*

$$(59)\qquad \|\theta-\tilde{\theta}\|_{1,2} \le M_{36} t_0^{\gamma_{12}} |||u-\tilde{u}|||, \quad \gamma_{12}>0$$

where $M_{36} = M_{36}(M_{12}, M_{13}, M_{14}, R_3)$.

Proof of Theorem 5: Using equation (2), we get for $z=\theta-\tilde{\theta}$

$$(60)\qquad z'_t - k_4[(1+u'_0)^{-1}z'_x]'_x = k_4[(\tilde{u}''_{xx}-u''_{xx})(1+u'_x)^{-1}(1+\tilde{u}'_x)\theta'_x + (\tilde{u}'_x-u'_x)(1+u'_x)^{-1}(1+\tilde{u}'_x)^{-1}\theta''_{xx} + (\tilde{u}'_x-u'_x)[(1+u'_x)^{-1}(1+\tilde{u}'_x)^{-1}]'_x\tilde{\theta}'_x] - k_3[(u''_{tx}-u''_{tx})\theta - \tilde{u}''_{tx}(\theta-\tilde{\theta})] - k_2(u''_{tx}-\tilde{u}''_{tx})(u''_{tx}+\tilde{u}''_{tx}) \equiv w;$$

$$(61)\qquad z(0,x)=0, \quad z'_t(0,x)=0; \quad z(t,0)=z(t,1)=0.$$

This implies in view of (9)

$$(62)\qquad \|\theta-\tilde{\theta}\|_{1,2} \le M_{37}\|w\|_0.$$

Let us show that for a certain γ_{13} the inequality

$$(63)\qquad \|w\|_0 \le M_{38}(t_0^{\gamma_{13}}|||u-\tilde{u}||| + t_0^{1-1/p}\|\theta-\tilde{\theta}\|_{1,2})$$

is valid.

For this we observe that if the function $v(t,x)$ satisfies the condition $v(0,x)=0$, then we have the inequalities

$$(64)\qquad \begin{aligned} \max_{(t,x)\in Q}|v(t,x)| &\le M_{39}t_0^{1-1/p};\\ \max_{(t,x)\in Q}|v'_x(t,x)| &\le M_{40}|||v|||t_0^{1-1/p};\\ \|v''_{xx}\|_0 &\le M_{41}|||v|||t_0^{1-1/p}. \end{aligned}$$

The proof of these inequalities is not difficult and carried out with the help of Newton–Leibnitz formula, Hölder inequality and Sobolev's embedding theorems. Using these inequalities, (56), (57) and estimating w defined by (61), we obtain inequality (62). It is easy to see that M_{38} depends on M_i $(i=12,13,14)$ and R_3.

The inequality (59) follows from (62) for sufficiently small t_0. Theorem 5 is proved. ∎

7.

Proof of Theorem 1: Let us consider the set

$$S_4 = \{\theta : \|\theta\|_{1,2} \le R_4, \theta(0,x) = \theta_0(x)\}$$

in $W_p^{1,2}$ as a complete metric space with the $W_p^{1,2}$ metric. Theorem 2 implies the existence of a unique solution u of the problem (1), (3) for any $\theta \in S_{R_4}$ and sufficiently small $t_0 = t_0'$. Theorem 4 in turn implies the existence of a unique solution $\tilde{\theta}$ of the problem (2), (4) for sufficiently small $t_0 = t_0''$ according to the properties of the solution u. Choosing $t_0 = \min(t_0', t_0'')$, we obtain that the operator $\tilde{\theta} = \mathfrak{N}(\theta)$ is defined on S_{R_4}.

Let us show that the operator $\mathfrak{N}$ has the unique fixed point in S_{R_4}. At first let us show that for sufficiently large R_4 and sufficiently small t_0 the inclusion

$$\mathfrak{N} S_{R_4} \subset S_{R_4} \tag{65}$$

is valid.

Taking into account (55) and the relation $\tilde{\theta}(0,x) = \theta_0(x)$, we get

$$\|\tilde{\theta}\|_{1,2} \le M_{31}(M_{12}, M_{13}, M_{14})(R_3 + 1).$$

Here the constants M_i $(i = 12, 13, 14)$ are taken from the inequalities (27)–(29) which are valid for the solution $u(t,x)$ of the problem (1), (3) for fixed $\theta \in S_{R_4}$.

In order to prove (65) it is sufficient to show that

$$M_{31}(R_3 + 1) \le R_4 \tag{66}$$

for any $\theta \in S_{R_4}$. In turn it is sufficient to show that M_{31} does not depend on R_4. If $\theta \in S_{R_4}$ and $3 < p < +\infty$, then from inequalities (9) and (10) it follows that $\theta_x' \in C[Q]$ and

$$\max_{(t,x)\in Q} |\theta_x'(t,x)| \le M_{42}(\|\theta\|_{1,2} + |\theta_0|_{2-2/p}).$$

This implies that

$$\|\theta_x'\|_0 \le M_{43}(\|\theta\|_{1,2} + |\theta_0|_{2-2/p}) t_0^{1/p} \le M_{43}(R_4 + R_0) t_0^{1/p}.$$

Choosing t_0 sufficiently small we get

$$\|\theta_x'\|_0 \le R_0$$

for fixed R_0 uniformly with respect to $\theta \in S_{R_4}$. From this and theorem 2 we have that the constants M_i ($i = 12, 13, 14$) (and M_{31} accordingly) does not increase for increasing R_4. Therefore, the inequality (66) is valid for sufficiently large R_4 (with smaller t_0 if necessary). The inequality (65) is established.

Let us show that the operator $\mathfrak{N}$ is a contraction in S_{R_4}. Let $\theta_1 = \mathfrak{N}(\theta), \tilde{\theta}_1 = \mathfrak{N}(\tilde{\theta})$ where $\theta, \tilde{\theta} \in S_{R_2}$ and $u, \tilde{u}$ be a solution of (1), (3) for θ and $\tilde{\theta}$. From Theorem 5 we get

$$\|\theta_1 - \tilde{\theta}_1\|_{1,2} \leq M_{36} t_0^{\gamma_{12}} |||u - \tilde{u}|||. \tag{67}$$

In turn from Theorem 3 it follows that

$$|||u - \tilde{u}||| \leq M_{43} \|\theta - \tilde{\theta}\|_{1,2}. \tag{68}$$

From (67) and (68) we obtain that $\mathfrak{N}$ is a contraction in S_{R_4} for sufficiently small t_0.

Now applying the fixed point theorem we get the existence of a unique fixed point θ of the operator $\mathfrak{N}$. It is easy to see that the pair θ, u (u is the solution of (1), (3) for this fixed point θ) is a unique solution of the problem (1)–(4). Theorem 1 is proved. ∎

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